Published by Institute of Physics Publishing for SISSA

Received: *September 13, 2006*
Accepted: *October 18, 2006*
Published: *November 27, 2006*

Dyon spectrum in $\mathcal{N} = 4$ supersymmetric type II string theories

Justin R. David, Dileep P. Jatkar and Ashoke Sen

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, India
E-mail: justin@mri.ernet.in, dileep@mri.ernet.in, sen@mri.ernet.in, ashoke.sen@cern.ch

Abstract: We compute the spectrum of quarter BPS dyons in freely acting $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds of type II string theory compactified on a six dimensional torus. For large charges the result for statistical entropy computed from the degeneracy formula agrees with the corresponding black hole entropy to first non-leading order after taking into account corrections due to the curvature squared terms in the effective action. The result is significant since in these theories the entropy of a small black hole, computed using the curvature squared corrections to the effective action, fails to reproduce the statistical entropy associated with elementary string states.

Keywords: Black Holes in String Theory, D-branes.

Contents

1. Introduction and summary

By now there is a reasonably good understanding of the spectrum of 1/4 BPS dyons in a class of $\mathcal{N} = 4$ supersymmetric string theories in four dimensons [1 − 9]. These include heterotic string theory on a torus as well as a class of CHL models [10 − 15] obtained by $\mathbb{Z}_N$ orbifolding of toroidally compactified heterotic string theory. Dual description of these theories involve type IIA string theory compactified on $K3 \times T^2$ and appropriate $\mathbb{Z}_N$ orbifolds of this theory. In each example studied so far, the statistical entropy computed by taking the logarithm of the degeneracy of states agrees with the entropy of the corresponding black hole for large charges, not only in the leading order but also in the first non-leading order [2, 6, 9]. On the black hole side this requires inclusion of four derivative terms in the effective action, and use of Wald's generalized formula for the black hole entropy in the presence of higher derivative corrections [16 − 19].

In this paper we extend this analysis to yet another $\mathcal{N} = 4$ supersymmetric string theory, obtained by taking a freely acting $\mathbb{Z}_2$ orbifold of type IIA string theory compactified on a six torus T^6. The orbifold group involves reflection of four coordinates of the torus together with half unit of shift along a fifth direction on the torus. There is a dual description of this model, also as an orbifold of type IIA string theory on T^6, but now the orbifold group involves half unit of shift along one coordinate of the torus together with a $(-1)^{F_L}$ transformation where F_L is the contribution to the space-time fermion number from the left-moving sector of the string world-sheet [20]. Although in many respects this model has very similar properties to the $\mathcal{N}=4$ supersymmetric heterotic string compactification studied earlier, there is one important difference. Unlike in the $\mathcal{N}=4$ theories coming from heterotic string compactification, in the present model the entropy of a small black hole representing an elementary string state fails to reproduce the statistical entropy associated with elementary string states [21, 22]. This makes it important to test if the statistical entropy of dyons agrees with the black hole entropy.

We follow the procedure of [9] to compute the degeneracy of a class of dyons in this theory. The result may be summarized as follows. Let us denote by Q_e and Q_m the electric and magnetic charge vectors of a state in the second description of the theory where the orbifold group involves a $(-1)^{F_L}$ transformation, and by $a\cdot b$ the T-duality invariant inner products between two such charge vectors a and b. Then the degeneracy $d(Q_e, Q_m)$ of a class of 1/4 BPS dyonic states are given by

$$d(Q_e, Q_m) = -\frac{1}{2^9}\int_C d\tilde\rho d\tilde\sigma d\tilde v \frac{1}{\tilde\Phi(\tilde\rho,\tilde\sigma,\tilde v)} \exp\left[-i\pi(2\tilde\rho Q_e^2 + \tilde\sigma Q_m^2/2 + 2\tilde v Q_e\cdot Q_m)\right], \tag{1.1}$$

where $Q_e^2 \equiv Q_e\cdot Q_e$, $Q_m^2 \equiv Q_m\cdot Q_m$, $\tilde\Phi$ is a function to be specified below, and C is a three real dimensional subspace of the three complex dimensional space labelled by $(\tilde\rho,\tilde\sigma,\tilde v)$, given by

$$\begin{aligned} &Im\,\tilde\rho = M_1, \quad Im\,\tilde\sigma = M_2, \quad Im\,\tilde v = M_3, \\ 0\le &Re\,\tilde\rho \le 1, \quad 0\le Re\,\tilde\sigma\le 2, \quad 0\le Re\,\tilde v \le 1\,, \end{aligned} \tag{1.2}$$

M_1, M_2 and M_3 being fixed large positive numbers. The function $\tilde\Phi$ is given by

$$\tilde\Phi(\tilde\rho,\tilde\sigma,\tilde v) = -\frac{1}{2^8}\, e^{2\pi i(\tilde\rho+\tilde v)}\prod_{r=0}^{1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{2},l,j\in\mathbb{Z}\\ k',l\ge 0, j<0 \text{ for } k'=l=0}} \left(1-e^{2\pi i(\tilde\sigma k'+\tilde\rho l+\tilde v j)}\right)^{\sum_{s=0}^{1}(-1)^{sl}c^{(r,s)}(4lk'-j^2)}, \tag{1.3}$$

where the coefficients $c^{(r,s)}(4lk'-j^2)$ are given as follows. Let us denote by $\tilde g$ a transformation that changes the sign of all the coordinates of a four torus T^4, and consider a (4,4) superconformal field theory (SCFT) with target space T^4. We now take an orbifold of this theory by the $\mathbb{Z}_2$ group generated by $\tilde g$, and define

$$F^{(r,s)}(\tau,z) \equiv \frac{1}{2}Tr_{RR;\tilde g^r}\left(\tilde g^s(-1)^{F_L+F_R}e^{2\pi i\tau L_0}e^{2\pi i\mathcal{J}z}\right), \qquad r,s=0,1\,, \tag{1.4}$$

where Tr denotes trace over all the Ramond-Ramond (RR) sector states twisted by $\tilde g^r$ in this SCFT before we project on to $\tilde g$ invariant states. F_L and F_R denote the world-sheet fermion numbers[1] associated with left and right chiral fermions in this SCFT, and $\mathcal{J}/2$ is the generator of the $\mathrm{U}(1)_L$ subgroup of the $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ R-symmetry group of this

[1] For the world-volume theory on the D-branes the world-volume fermion number coincides with the space-time fermion number. For describing elementary string states we shall mostly use light-cone gauge Green-Schwarz formalism where again the world-sheet fermion number coincides with the space-time fermion number. Thus throughout this paper there will be no distinction between world-sheet and space-time fermion numbers.

conformal field theory. One finds that $F^{(r,s)}(\tau,z)$ has expansion of the form

$$F^{(r,s)}(\tau,z)=\sum_{b\in\mathbb{Z},n} c^{(r,s)}(4n-b^2)e^{2\pi in\tau+2\pi ibz}\,. \tag{1.5}$$

This defines the coefficients $c^{(r,s)}(u)$.

The explicit forms of $F^{(r,s)}(\tau,z)$ are as follows

$$\begin{aligned} F^{(0,0)}(\tau,z) &= 0 \\ F^{(0,1)}(\tau,z) &= 8\,\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} \\ F^{(1,0)}(\tau,z) &= 8\,\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \\ F^{(1,1)}(\tau,z) &= 8\,\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}\,. \end{aligned} \tag{1.6}$$

From this one can calculate the coefficients $c^{(r,s)}(u)$ explicitly.

As in the case of CHL models, the function $\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})$ turns out to be a modular form of weight 2 under a certain subgroup of the Siegel modular group of genus two Riemann surfaces. Using this fact one can prove that the degeneracy formula (1.1) is invariant under the S-duality group $\Gamma_1(2)$ of the theory.

Using (1.1) one can also compute the statistical entropy of the dyon for large charges following the general strategy outlined in [2, 6, 9] and compare it with the entropy of the corresponding black hole. It turns out that up to order Q^0 both the statistical entropy and the black hole entropy are obtained by extremizing the function

$$\frac{\pi}{2\tau_2}\,|Q_e+\tau Q_m|^2-\ln\tilde{f}(\tau)-\ln\tilde{f}(-\bar{\tau})-4\ln(2\tau_2)+\text{constant}+\mathcal{O}(Q^{-2})\,, \tag{1.7}$$

with respect to the real and imaginary parts of $\tau=\tau_1+i\tau_2$. Here

$$\tilde{f}(\tau)=\eta(\tau)^{16}/\eta(2\tau)^8\,. \tag{1.8}$$

Thus we see that to this order the black hole entropy agrees with the statistical entropy. The result is significant in light of the fact that the same four derivative corrections to the effective action fail to reproduce the statistical entropy of elementary string states in this theory, essentially due to the fact that these corrections vanish at the tree level.

These results can also be generalized to a freely acting $\mathbb{Z}_3$ orbifold of type II string theory compactified on a six dimensional torus. For brevity we shall not give the results here, but a summary of the results can be found in section 6.

The rest of the paper is organised as follows. In section 2 we describe the theory under consideration in different duality frames, and also describe the dyon configuration that we shall analyze in this paper. In section 3 we count the degeneracy of a class of 1/4 BPS dyonic states with a given set of charges, and reproduce eq. (1.1). In section 4 we use the techniques developed in [7] to show that $\tilde{\Phi}$ transforms as a modular form under a subgroup of $\mathrm{Sp}(2,\mathbb{Z})$. This in turn proves the S-duality invariance of (1.1). In section 5 we analyze

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the behaviour of the statistical entropy computed from (1.1) for large charges and show that it agrees with the black hole entropy up to first non-leading order. Section 6 contains a summary of the results for the $\mathbb{Z}_3$ orbifold theory.

Since most of the analysis in this paper is identical to that in [7, 9] we often skip the details of the calculation and quote the final result. For details of the calculation the reader should consult the original references.

2. The dyon configuration

In this section we shall describe the model under consideration and its various dual descriptions which will be relevant for our analysis. The analysis is based on the connection between four and five dimensional black holes discussed in [23, 3, 9].

1. We begin with type IIB string theory compactified on a six torus $T^4 \times S^1 \times \tilde{S}^1$, and take a system containing Q_5 D5-branes wrapped on $T^4 \times S^1$, Q_1 D1-branes wrapped on S^1, $-n$ units of momentum along S^1, J units of momentum along $\tilde{S}^1$ and a Kaluza-Klein monopole associated with the compact circle $\tilde{S}^1$. For definiteness we shall label S^1 and $\tilde{S}^1$ by coordinates with period 2π. Let us denote the coordinates of T^4 by x^6, x^7, x^8, x^9, and the coordinates of $\tilde{S}^1$, S^1 by x^4, x^5. We then take an orbifold of this system by a $\mathbb{Z}_2$ transformation generated by
$$g : (x^4, x^5, x^6, x^7, x^8, x^9) \to (x^4, x^5 + \pi, -x^6, -x^7, -x^8, -x^9)\,. \qquad (2.1)$$
We shall denote by $\tilde{g}$ the part of g that acts on T^4, i.e.
$$\tilde{g} : (x^4, x^5, x^6, x^7, x^8, x^9) \to (x^4, x^5, -x^6, -x^7, -x^8, -x^9)\,. \qquad (2.2)$$
We shall call this the first description of the system.

2. We now make an S-duality transformation on this system to get type IIB string theory on $T^4 \times S^1 \times \tilde{S}^1/\mathbb{Z}_2$ with Q_5 NS5-branes on $T^4 \times S^1$, Q_1 units of fundamental string winding charge along S^1, $-n$ units of momentum along S^1, J units of momentum along $\tilde{S}^1$, and a Kaluza-Klein monopole associated with $\tilde{S}^1$ compactification. Under this duality the generators g and $\tilde{g}$ remain unchanged.

3. Next make an $R \to 1/R$ duality transformation along $\tilde{S}^1$ to convert the theory into type IIA string theory on $T^4 \times S^1 \times \hat{S}^1/\mathbb{Z}_2$ with Q_5 Kaluza-Klein monopoles associated with $\hat{S}^1$ compactification, Q_1 units of fundamental string winding charge along S^1, $-n$ units of momentum along S^1, J units of fundamental string winding charge along $\hat{S}^1$, and a single NS5-brane wrapped on $T^4 \times S^1$. Here $\hat{S}^1$ denotes the dual circle of $\tilde{S}^1$. Again the generators g and $\tilde{g}$ remain unchanged under this duality transformation.

4. Finally using the string-string self-duality described in [20] we can relate this to a type IIA string theory on $\hat{T}^4 \times S^1 \times \hat{S}^1/\mathbb{Z}_2'$, where the generator of $\mathbb{Z}_2'$ involves half unit of shift along S^1 together with a $(-1)^{F_L}$ transformation where F_L denotes the contribution to the space-time fermion number from the left-moving sector of the

string world-sheet. The action of this duality on various states is similar to that of string-string duality relating type IIA string theory on K3 and heterotic string theory on T^4. The final system consists of Q_5 Kaluza-Klein monopoles associated with $\hat{S}^1$ compactification, Q_1 units of NS5-brane charge along $\hat{T}^4 \times S^1$, $-n$ units of momentum along S^1, J units of NS5-brane charge along $\hat{T}^4 \times \hat{S}^1$, and a single fundamental string wrapped on S^1. We shall call this description the second description of the system.

Since the second description has only fundamental strings, NS 5-branes and Kaluza-Klein monopoles, we shall use this description to identify the various charges as electric or magnetic. If $-\vec{n}$ and $\vec{w}$ denote the momentum and winding charges respectively along $S^1 \times \hat{S}^1$, and $\vec{N}$ and $\vec{W}$ denote the Kaluza-Klein monopole charges and H-monopole charges (NS-5-branes transverse to the circle) along $S^1 \times \hat{S}^1$, then we can define the T-duality invariant inner product

$$Q_e^2 = 2\vec{n} \cdot \vec{w}, \quad Q_m^2 = 2\vec{N} \cdot \vec{W}, \quad Q_e \cdot Q_m = \vec{n} \cdot \vec{N} + \vec{w} \cdot \vec{W}\,. \tag{2.3}$$

Thus before the $\mathbb{Z}_2$ modding we had $\frac{1}{2}Q_m^2 = Q_1 Q_5$, $\frac{1}{2}Q_e^2 = n$, and $Q_e.Q_m = J$. In order to get a $\mathbb{Z}_2$ invariant configuration so that we can carry out the $\mathbb{Z}_2$ modding, we need to put periodic boundary conditions on all the branes which extend along S^1, and take 2 identical copies of all the branes transverse to S^1 and place them at intervals of π along S^1. The latter set includes the five branes along $\hat{T}^4 \times \hat{S}^1$; we need to take $2J$ five branes, divide them into two sets and place the two sets separated by an interval of π along S^1. After orbifolding the direction along S^1 can be regarded as a circle of radius 1/2, and per unit period along S^1 there will be J five branes transverse to S^1. The natural unit of momentum along S^1 is now 2, and momentum $-n$ along S^1 can be regarded as $-n/2$ units of momentum. The other charges have the same values as in the parent theory. Thus we now have

$$\frac{1}{2}Q_e^2 = n/2, \qquad \frac{1}{2}Q_m^2 = Q_1\,Q_5, \qquad Q_e \cdot Q_m = J\,. \tag{2.4}$$

Before concluding this section we shall make a few remarks about the supersymmetry and S-duality symmetry of the theory and also the spectrum of massless states in the theory. Type II string theory compactified on torus has 32 supercharges, but the $\mathbb{Z}_2$ orbifolding breaks half of these supersymmetries. In the first description half of the supersymmetries from the left-moving sector of the world-sheet and half of the supersymmetries from the right-moving sector of the world-sheet are broken. Thus this description is analogous to type II string theory compactified on $K3 \times T^2$. In the second description all the supersymmetries from the left-moving sector of the world-sheet are broken and all supersymmetries from the right-moving sector of the world-sheet are preserved. Thus this situation is analogous to heterotic string theory on T^6. As in [9] the dyon system breaks 3/4 of the supersymmetry generators; hence these are 1/4 BPS states of the theory.

The S-duality symmetry of this theory in the second description may be analysed by mapping it to the T-duality symmetry of the theory in the first description. It is essentially the subgroup of the T-duality symmetry SL$(2, \mathbb{Z})$ of T^2 that commutes with half unit of shift along S^1, and is generated by the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying

$$ad - bc = 1, \quad a, d \in 1 + 2\mathbb{Z}, \quad c \in 2\mathbb{Z}, \quad b \in \mathbb{Z}. \tag{2.5}$$

This defines the group $\Gamma_1(2) \equiv \Gamma_0(2)$ [13].

The spectrum of massless states may be analyzed easily using the second description of the theory. First of all since the theory has $\mathcal{N} = 4$ supersymmetry, the low energy effective field theory must be $\mathcal{N} = 4$ supergravity coupled to a set of matter multiplets. Thus in order to find the spectrum all we need to do is to find the number of matter multiplets. This in turn is equal to the number of massless vector fields (the rank of the gauge group) minus six, since there are six graviphotons. To count the number of massless vector fields we note that since in the Neveu-Schwarz-Ramond (NSR) formulation the $\mathbb{Z}_2$ transformation changes the sign of all the Ramond (R) sector states on the left, it projects out all the massless states (including the gauge fields) originating in the RR sector. On the other hand since it acts trivially on the massless NS-NS sector states, all the 12 gauge fields in the NS-NS sector coming from the components of the metric and rank two anti-symmetric tensor fields along the internal directions of the torus survive the projection. This gives a rank 12 gauge group. Thus we have six matter multiplets.

3. Counting of states of the dyon

The description of the system given in the previous section makes it clear that the system is very similar to the corresponding system in the $\mathbb{Z}_2$ CHL model analyzed in [9] with $K3$ replaced by T^4, and the transformation $\tilde{g}$ given by (2.2) rather than a $\mathbb{Z}_2$ involution in K3. Thus the computation of the degeneracy proceeds in a manner identical to that in [9]. We now outline the main steps in this computation.

As shown in [9] the final result for the degeneracy depends only on the combination $Q_1 Q_5$; hence we shall for simplicity consider the $Q_5 = 1$ case.[2] In the first description of the system the quantum numbers n and J arise from three different sources: the excitations of the Kaluza-Klein monopole which can carry certain amount of momentum $-l_0'$ along S^1, the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole which can carry certain amount of momentum $-l_0$ along S^1 and j_0 along $\tilde{S}^1$ and the motion of the D1-branes in the plane of the D5-brane carrying total momentum $-L$ along S^1 and J' along $\tilde{S}^1$. Thus we have

$$l_0' + l_0 + L = n, \qquad j_0 + J' = J. \tag{3.1}$$

Let $h(Q_1, n, J)$ denote the number of bosonic minus fermionic supermultiplets (in the sense described in [9]) of the combined system carrying quantum numbers Q_1, n, J and let

$$f(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \sum_{Q_1, n, J} h(Q_1, n, J) e^{2\pi i (\tilde{\rho} n + \tilde{\sigma} Q_1/2 + \tilde{v} J)}, \tag{3.2}$$

denote the partition function of the system. Then $f(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ is obtained as a product of

[2] Unlike in [9] where wrapping a D5-brane on $K3$ shifted the Q_1 charge by $-Q_5$, a D5-brane wrapped on T^4 does not cause any such shift.

three separate partition functions:

$$f(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \frac{1}{64} \sum_{Q_1, L, J'} d_{D1}(Q_1, L, J') e^{2\pi i(\tilde{\sigma} Q_1/2 + \tilde{\rho} L + \tilde{v} J')}$$
$$\left(\sum_{l_0, j_0} d_{CM}(l_0, j_0) e^{2\pi i l_0 \tilde{\rho} + 2\pi i j_0 \tilde{v}} \right) \left(\sum_{l'_0} d_{KK}(l'_0) e^{2\pi i l'_0 \tilde{\rho}} \right) , \qquad (3.3)$$

where $d_{D1}(Q_1, L, J')$ is the degeneracy of Q_1 D1-branes moving in the plane of the D5-brane carrying momenta $(-L, J')$ along $(S^1, \tilde{S}^1)$, $d_{CM}(l_0, j_0)$ is the degeneracy associated with the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole carrying momenta $(-l_0, j_0)$ along $(S^1, \tilde{S}^1)$ and $d_{KK}(l'_0)$ denotes the degeneracy associated with the excitations of a Kaluza-Klein monopole carrying momentum $-l'_0$ along S^1. The factor of 1/64 in (3.3) accounts for the fact that a single 1/4 BPS supermultiplet has 64 states.

We begin with the computation of $d_{KK}(l'_0)$. Under the duality that relates the first description to the second description, a Kaluza-Klein monopole in the first description gets mapped to a twisted sector fundamental string in the second description, and the transformation $\tilde{g}$ gets mapped to $\hat{g} = (-1)^{F_L}$. Let us consider a (4,4) superconformal field theory describing type IIA string theory compactified on $T^4 \times S^1 \times \hat{S}^1$ in the light-cone gauge Green-Schwarz formalism. Following the procedure of [9] one finds that

$$\sum_{l'_0} d_{KK}(l'_0) e^{2\pi i l'_0 \tilde{\rho}} = Tr'_{\hat{g}} \left((-1)^{F_L} e^{4\pi i \rho L'_0} \right) , \qquad (3.4)$$

where $Tr'_{\hat{g}}$ denotes trace over states for which the right-moving oscillators are in their ground state, and the left-moving oscillators are twisted by $\hat{g}$. We do not impose the requirement of $\hat{g}$ invariance on the states while taking the trace [9]. The factor of $(-1)^{F_L}$ inside the trace accounts for the fact that we want to count bosonic and fermionic excitations in the left-moving sector of the world-sheet with weights 1 and -1 respectively. This factor was not present in the corresponding expression in [9] since all the left-moving world-sheet oscillators were bosonic. The Virasoro generator L'_0 includes the contribution from all the left moving bosonic and fermionic oscillators but not from momenta or winding charges which are set to some fixed values. Since in the Green-Schwarz formulation there are 8 left-moving bosonic oscillators with periodic boundary condition and 8 left-moving fermionic oscillators with anti-periodic boundary condition (due to twisting by $\hat{g}$ under which the fermions are odd) we get

$$\sum_{l'_0} d_{KK}(l'_0) e^{2\pi i l'_0 \tilde{\rho}} = 16\, e^{-2\pi i \tilde{\rho}} \frac{\prod_{n=1}^{\infty} (1 - e^{2\pi i (2n-1)\tilde{\rho}})^8}{\prod_{n=1}^{\infty} (1 - e^{4\pi i n \tilde{\rho}})^8} = 16\, \frac{\eta(\tilde{\rho})^8}{\eta(2\tilde{\rho})^{16}} \, . \qquad (3.5)$$

The factor of 16 comes from the fermionic zero mode quantization in the right-moving sector. The overall factor of $e^{-2\pi i \tilde{\rho}}$ reflects the effect of the zero point energy.

Next we compute $d_{CM}(l_0, j_0)$. In this case besides the degrees of freedom associated with the motion of the D1-D5 system transverse to the plane of the D5-brane as in [9], there is an additional set of degrees of freedom associated with the Wilson lines along T^4 on the D5-brane [24]. This gives rise to four additional bosonic fields together with their fermionic superpartners living on S^1. For the degeneracy associated with the dynamics transverse to the plane of the D5-brane, not only the computational procedure but also the results are identical to that in [9] for the $\mathbb{Z}_2$ orbifold case, and we get

$$\sum_{l_0,j_0} d_{transverse}(l_0,j_0) e^{2\pi i l_0\tilde{\rho}+2\pi i j_0\tilde{v}} = 4\, e^{-2\pi i\tilde{v}}\,(1-e^{-2\pi i\tilde{v}})^{-2} \\ \prod_{n=1}^{\infty}\left\{(1-e^{4\pi i n\tilde{\rho}})^4\,(1-e^{4\pi i n\tilde{\rho}+2\pi i\tilde{v}})^{-2}\,(1-e^{4\pi i n\tilde{\rho}-2\pi i\tilde{v}})^{-2}\right\}. \qquad (3.6)$$

On the other hand the bosonic fields associated with the Wilson line along T^4 and their fermionic superpartners are odd under $\tilde{g}$, and hence have anti-periodic boundary condition along S^1. Together they describe a (4,4) superconformal field theory with $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ R-symmetry, and the quantum number j_0 may be identified with twice the eigenvalue of the $\mathrm{U}(1)_L$ generator of $\mathrm{SU}(2)_L$ [25]. The bosons and the right-moving fermions are neutral under $\mathrm{SU}(2)_L$ and hence do not carry any j_0 quantum number, but the left-moving fermions are doublets under the $\mathrm{SU}(2)_L$ R-symmetry group and hence carry j_0 quantum numbers ± 1.[3] Thus we have

$$\sum_{l_0,j_0} d_{wilson}(l_0,j_0) e^{2\pi i l_0\tilde{\rho}+2\pi i j_0\tilde{v}} = \prod_{n=1}^{\infty}\Big\{(1-e^{2\pi i(2n-1)\tilde{\rho}})^{-4}\,(1-e^{2\pi i(2n-1)\tilde{\rho}+2\pi i\tilde{v}})^2 \\ (1-e^{2\pi i(2n-1)\tilde{\rho}-2\pi i\tilde{v}})^2\Big\}. \qquad (3.7)$$

The partition function associated with $d_{CM}(l_0,j_0)$ is given by the product of these two contributions:

$$\sum_{l_0,j_0} d_{CM}(l_0,j_0) e^{2\pi i l_0\tilde{\rho}+2\pi i j_0\tilde{v}} = 4\, e^{-2\pi i\tilde{v}}\,(1-e^{-2\pi i\tilde{v}})^{-2} \\ \prod_{n=1}^{\infty}\left\{(1-e^{4\pi i n\tilde{\rho}})^4\,(1-e^{4\pi i n\tilde{\rho}+2\pi i\tilde{v}})^{-2}\,(1-e^{4\pi i n\tilde{\rho}-2\pi i\tilde{v}})^{-2}\right\} \\ \prod_{n=1}^{\infty}\Big\{(1-e^{2\pi i(2n-1)\tilde{\rho}})^{-4}\,(1-e^{2\pi i(2n-1)\tilde{\rho}+2\pi i\tilde{v}})^2\,(1-e^{2\pi i(2n-1)\tilde{\rho}-2\pi i\tilde{v}})^2\Big\}. \qquad (3.8)$$

Finally we need to find $d_{D1}(Q_1,L,J')$. Since the analysis is identical to the one given in [26, 9], we shall only quote the result. We first define

$$F^{(r,s)}(\tau,z) \equiv \frac{1}{2} Tr_{RR;\tilde{g}^r}\left(\tilde{g}^s(-1)^{F_L+F_R} e^{2\pi i\tau L_0} e^{2\pi i\mathcal{J}z}\right), \qquad r,s=0,1, \qquad (3.9)$$

[3] Recall that the fermions which are superpartners of the bosonic fields representing transverse motion of the D-brane have exactly opposite properties. The left-moving fermions are neutral under $\mathrm{SU}(2)_L$ and the right-moving fermions transform in the doublet representation of $\mathrm{SU}(2)_L$ [9].

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where the trace is taken over all the RR sector states twisted by $\tilde{g}^r$ in a (4,4) superconformal field theory with target space $T^4/\mathbb{Z}_2$, – with $\mathbb{Z}_2$ generated by $\tilde{g}$, — before we project on to $\tilde{g}$ invariant states. F_L and F_R denote the world-sheet fermion numbers associated with left and right chiral fermions, and $\mathcal{J}/2$ is the generator of the $\mathrm{U}(1)_L$ subgroup of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-symmetry group of this conformal field theory. Explicit computation gives

$$\begin{aligned} F^{(0,0)}(\tau, z) &= 0 \\ F^{(0,1)}(\tau, z) &= 8\, \frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} \\ F^{(1,0)}(\tau, z) &= 8\, \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \\ F^{(1,1)}(\tau, z) &= 8\, \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2}\,. \end{aligned} \tag{3.10}$$

These can be rewritten as

$$F^{(r,s)}(\tau, z) = h_0^{(r,s)}(\tau)\, \vartheta_3(2\tau, 2z) + h_1^{(r,s)}(\tau)\, \vartheta_2(2\tau, 2z) \tag{3.11}$$

where

$$\begin{aligned} h_0^{(0,0)}(\tau) &= 0, \qquad h_1^{(0,0)}(\tau) = 0, \\ h_0^{(0,1)}(\tau) &= 4\frac{1}{\vartheta_3(2\tau, 0)}, \qquad h_1^{(0,1)}(\tau) = 4\frac{1}{\vartheta_2(2\tau, 0)}, \\ h_0^{(1,0)}(\tau) &= 8\frac{\vartheta_3(2\tau, 0)}{\vartheta_4(\tau, 0)^2}, \quad h_1^{(1,0)}(\tau) = -8\frac{\vartheta_2(2\tau, 0)}{\vartheta_4(\tau, 0)^2}, \\ h_0^{(1,1)}(\tau) &= 8\frac{\vartheta_3(2\tau, 0)}{\vartheta_3(\tau, 0)^2}, \quad h_1^{(1,1)}(\tau) = 8\frac{\vartheta_2(2\tau, 0)}{\vartheta_3(\tau, 0)^2}. \end{aligned} \tag{3.12}$$

We now define the coefficients $c^{(r,s)}(u)$ through the expansions

$$h_0^{(r,s)}(\tau) = \sum_n c^{(r,s)}(4n) q^n, \qquad h_1^{(r,s)}(\tau) = \sum_n c^{(r,s)}(4n) q^n\,. \tag{3.13}$$

From (3.12) we see that in the expansion of $h_l^{(r,s)}$, $n \in \mathbb{Z} - \frac{l}{4}$ for $r = 0$ and $n \in \frac{1}{2}\mathbb{Z} - \frac{l}{4}$ for $r = 1$. Thus for given (r, s) the $c^{(r,s)}(u)$ defined through the two equations in (3.13) have non-overlapping set of arguments. Substituting (3.13) into (3.11) and using the Fourier expansions of $\vartheta_3(2\tau, 2z)$, $\vartheta_2(2\tau, 2z)$ we get

$$F^{(r,s)}(\tau, z) = \sum_{b \in \mathbb{Z}, n} c^{(r,s)}(4n - b^2) e^{2\pi i n\tau + 2\pi i b z}\,. \tag{3.14}$$

Following the analysis of [9] one can show that

$$\sum_{Q_1, L, J'} d_{D1}(Q_1, L, J') e^{2\pi i(\tilde{\sigma} Q_1/2 + \tilde{\rho} L + \tilde{v} J')} = \prod_{\substack{w, l, j \in \mathbb{Z} \\ w > 0, l \geq 0}} \left(1 - e^{2\pi i(\tilde{\sigma} w/2 + \tilde{\rho} l + \tilde{v} j)}\right)^{-n(w, l, j)}, \tag{3.15}$$

where

$$n(w,l,j) = \sum_{s=0}^{1} (-1)^{sl} c^{(r,s)}(2lw - j^2)\,, \qquad r = w \bmod 2\,. \tag{3.16}$$

It is now time to put the results together. Using the results

$$c^{(0,0)}(0) = 0\,, \qquad c^{(0,0)}(-1) = 0\,, \qquad c^{(0,1)}(0) = 4\,, \qquad c^{(0,1)}(-1) = 2\,, \tag{3.17}$$

and eqs. (3.3)), ((3.5), (3.8) and (3.15) we get

$$f(\tilde\rho, \tilde\sigma, \tilde v) = e^{-2\pi i(\tilde\rho + \tilde v)} \prod_{r=0}^{1} \prod_{\substack{k' \in \mathbb{Z} + \frac{r}{2}, l, j \in \mathbb{Z} \\ k', l \geq 0, j<0 \text{ for } k'=l=0}} \left(1 - e^{2\pi i(\tilde\sigma k' + \tilde\rho l + \tilde v j)}\right)^{-\sum_{s=0}^{1} (-1)^{sl}\, c^{(r,s)}(4lk' - j^2)}\,. \tag{3.18}$$

Defining

$$\tilde\Phi(\tilde\rho, \tilde\sigma, \tilde v) = -\frac{1}{2^8}\, e^{2\pi i(\tilde\rho + \tilde v)} \prod_{r=0}^{1} \prod_{\substack{k' \in \mathbb{Z} + \frac{r}{2}, l, j \in \mathbb{Z} \\ k', l \geq 0, j<0 \text{ for } k'=l=0}} \left(1 - e^{2\pi i(\tilde\sigma k' + \tilde\rho l + \tilde v j)}\right)^{\sum_{s=0}^{1} (-1)^{sl} c^{(r,s)}(4lk' - j^2)}\,, \tag{3.19}$$

we can express (3.18) as

$$f(\tilde\rho, \tilde\sigma, \tilde v) = -\frac{1}{2^8\, \tilde\Phi(\tilde\rho, \tilde\sigma, \tilde v)}\,. \tag{3.20}$$

Using (3.2) and identifying $h(Q_1, n, J)$ with the dyonic degeneracy $d(Q_e, Q_m)$ with $Q_e^2 = n$, $Q_m^2 = 2Q_1$ and $Q_e \cdot Q_m = J$, we get

$$d(Q_e, Q_m) \,=\, K \int_C d\tilde\rho d\tilde\sigma d\tilde v \frac{1}{\tilde\Phi(\tilde\rho, \tilde\sigma, \tilde v)} \exp\left[-i\pi(2\tilde\rho Q_e^2 + \tilde\sigma Q_m^2/2 + 2\tilde v Q_e \cdot Q_m)\right]\,, \tag{3.21}$$

where

$$K = -\frac{1}{2^9}\,, \tag{3.22}$$

and C denotes the surface

$$\begin{gathered} Im\, \tilde\rho = M_1, \quad Im\, \tilde\sigma = M_2, \quad Im\, \tilde v = M_3, \\ 0 \leq Re\, \tilde\rho \leq 1, \quad 0 \leq Re\, \tilde\sigma \leq 2, \quad 0 \leq Re\, \tilde v \leq 1\,, \end{gathered} \tag{3.23}$$

M_1, M_2, M_3 being fixed large positive numbers.

4. Properties of $\tilde\Phi$ from the threshold integral

In this section we shall derive various useful properties of $\tilde\Phi$, e.g. its duality transformation laws and locations of its zeroes by following the strategy described in [7, 9] for CHL models. The main idea is to begin with an integral that is manifestly invariant under a subgroup of the modular group Sp$(2, \mathbb{Z})$ of genus two Riemann surface and then express this as a sum of a holomorphic piece proportional to $\ln \tilde\Phi$, its complex conjugate and a piece that is neither holomorphic nor anti-holomorphic but has simple transformation properties under Sp$(2, \mathbb{Z})$ duality transformation. This in turn would determine the modular transformation laws of the holomorphic and the anti-holomorphic pieces separately.

4.1 The threshold integral

We define as in [7]

$$
\begin{aligned}
F_{m_1,m_2,n_1,n_2}(\tau,z) &= \sum_{s=0}^{1} (-1)^{m_1\, s}\, F^{(r,s)}(\tau,z) \quad \text{for } m_1,m_2,n_2 \in \mathbb{Z},\ n_1 \in \mathbb{Z}+\tfrac{r}{2},\ r=0,1 \\
&\equiv \sum_b F_{m_1,n_1,m_2,n_2;b}(\tau)\, e^{2\pi i b z}
\end{aligned} \tag{4.1}
$$

and

$$
\mathcal{I}(\tilde\rho,\tilde\sigma,\tilde v) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{\substack{m_1,m_2,n_2,b\in\mathbb{Z}\\ n_1\in\frac{1}{2}\mathbb{Z}}} q^{p_L^2/2-b^2/4}\bar q^{p_R^2/2} F_{m_1,m_2,n_1,n_2;b}(\tau) \tag{4.2}
$$

where $\mathcal{F}$ denotes the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ in the upper half plane, $F^{(r,s)}(\tau,z)$ have been defined in (3.9), and

$$
q = e^{2\pi i\tau}\,, \tag{4.3}
$$

$$
\begin{aligned}
\frac{1}{2}p_R^2 &= \frac{1}{4\det Im\Omega}|-m_1\tilde\rho + m_2 + n_1\tilde\sigma + n_2(\tilde\sigma\tilde\rho-\tilde v^2) + b\tilde v|^2, \\
\frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2 + m_1n_1 + m_2n_2 + \frac{1}{4}b^2\,,
\end{aligned} \tag{4.4}
$$

$$
\Omega = \begin{pmatrix} \tilde\rho & \tilde v \\ \tilde v & \tilde\sigma \end{pmatrix}. \tag{4.5}
$$

Using (3.11) the integral in (4.2) can be written as

$$
\mathcal{I}(\tilde\rho,\tilde\sigma,\tilde v) = \sum_{l,r,s=0}^{1} \mathcal{I}_{r,s,l} \tag{4.6}
$$

$$
\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z}\\ n_1\in\mathbb{Z}+\frac{r}{2},\, b\in 2\mathbb{Z}+l}} q^{p_L^2/2}\bar q^{p_R^2/2}(-1)^{m_1 s} h_l^{(r,s)}(\tau)\,. \tag{4.7}
$$

These integrals can be evaluated following the procedure of [27, 28, 7] by separately evaluating the contribution from the zero orbit, the degenerate orbits and the non-degenerate orbits. The only difference in the result from that in [7] arises from the fact that the coefficients $c^{(r,s)}(4n-b^2)$ now have different values. The final result is given by:

$$
\begin{aligned}
\mathcal{I}(\tilde\rho,\tilde\sigma,\tilde v) &= -2\ln\Bigg[\kappa(\det \mathrm{Im}\Omega)^2 \Bigg| \exp(2\pi i(\tilde\rho+\tilde v)) \\
&\qquad \prod_{r,s=0}^{1} \prod_{\substack{(l,b)\in\mathbb{Z},\,k'\in\mathbb{Z}+\frac{r}{2}\\ k',l\geq 0,\, b<0 \text{ for } k'=l=0}} \Big\{(1-\exp(2\pi i(k'\tilde\sigma + l\tilde\rho + b\tilde v)))^{(-1)^{ls}c^{(r,s)}(4k'l-b^2)}\Big\} \Bigg|^2\Bigg] \\
&= -2\ln\left[2^{16}\kappa(\det \mathrm{Im}\Omega)^2\right] - 2\ln\tilde\Phi(\tilde\rho,\tilde\sigma,\tilde v) - 2\ln\bar{\tilde\Phi}(\tilde\rho,\tilde\sigma,\tilde v)
\end{aligned} \tag{4.8}
$$

where $\tilde{\Phi}$ has been defined in (3.19) and

$$\kappa = \left(\frac{8\pi}{3\sqrt{3}} e^{1-\gamma_E} \right)^2 . \tag{4.9}$$

In arriving at (4.8) we have used

$$\begin{array}{llll} c^{(0,0)}(0) = 0, & c^{(0,0)}(-1) = 0, & c^{(0,1)}(0) = 4, & c^{(0,1)}(-1) = 2, \\ c^{(1,0)}(0) = 8, & c^{(1,0)}(-1) = 0, & c^{(1,1)}(0) = 8, & c^{(1,1)}(-1) = 0. \end{array} \tag{4.10}$$

Another useful integral is

$$\mathcal{I}'(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \mathcal{I}\left(\tilde{\rho} - \frac{\tilde{v}^2}{\tilde{\sigma}}, -\frac{1}{\tilde{\sigma}}, \frac{\tilde{v}}{\tilde{\sigma}} \right) . \tag{4.11}$$

By manipulating the expression for $\mathcal{I}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ given in (4.2) and the duality transformation properties of p_L^2 and p_R^2 one can show that [7]

$$\mathcal{I}'(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \sum_{l,r,s=0}^{1} \mathcal{I}'_{r,s,l} \tag{4.12}$$

$$\mathcal{I}'_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{\substack{m_1, n_1, n_2 \in \mathbb{Z} \\ m_2 \in \mathbb{Z} + \frac{r}{2}, b \in 2\mathbb{Z}+l}} q^{p_L^2/2} \bar{q}^{p_R^2/2} (-1)^{n_2 s} h_l^{(r,s)}(\tau) . \tag{4.13}$$

These integrals may also be analyzed following the procedure described in [7] and the result is

$$\mathcal{I}'(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = -2 \ln \left[2^{16} \kappa (\det \mathrm{Im}\Omega)^2 \right] - 2 \ln \Phi(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) - 2 \ln \bar{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) , \tag{4.14}$$

where

$$\begin{aligned} \Phi(\rho, \sigma, v) &= -\exp(2\pi i(\sigma + \rho + v)) \\ &\quad \prod_{r,s=0}^{1} \prod_{\substack{(k',l,b) \in \mathbb{Z} \\ k',l \geq 0, b<0 \text{ for } k'=l=0}} \left\{ 1 - (-1)^r \exp(2\pi i(k'\sigma + l\rho + bv) \right\}^{c^{(r,s)}(4k'l - b^2)} . \end{aligned} \tag{4.15}$$

It follows from (4.8), (4.14) and the relation (4.11) between $\mathcal{I}$ and $\mathcal{I}'$ that[4]

$$\Phi(\rho, \sigma, v) = \sigma^{-2} \, \tilde{\Phi}\left(\rho - \frac{v^2}{\sigma}, -\frac{1}{\sigma}, \frac{v}{\sigma} \right) , \quad \tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \tilde{\sigma}^{-2} \, \Phi\left(\tilde{\rho} - \frac{\tilde{v}^2}{\tilde{\sigma}}, -\frac{1}{\tilde{\sigma}}, \frac{\tilde{v}}{\tilde{\sigma}} \right) . \tag{4.16}$$

We shall now use these relations to analyze various properties of $\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$.

[4] This analysis does not determine the relative phase between Φ and $\tilde{\Phi}$. This can be fixed by comparing the $v \to 0$ (or $\tilde{v} \to 0$) limit of the two sides of eq. (4.16).

4.2 Duality transformation properties

Following the same line of argument as in [7] for the $\mathbb{Z}_2$ CHL model, the original integral $\mathcal{I}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ can be shown to be invariant under a transformation:

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}\,, \tag{4.17}$$

if the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belongs to a subgroup $\tilde{G}$ of $\mathrm{Sp}(2, \mathbb{Z})$ defined in [6]. Using the invariance of $\mathcal{I}$ under (4.17) and the relation (4.8) we see that $\tilde{\Phi}$ is a modular form of weight 2 under the subgroup $\tilde{G}$ of $\mathrm{Sp}(2, \mathbb{Z})$:

$$\tilde{\Phi}\left((A\Omega + B)(C\Omega + D)^{-1}\right) = \det(C\Omega + D)^2 \tilde{\Phi}(\Omega), \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{G}\,. \tag{4.18}$$

Using this result we can now follow the procedure of [6] to establish the invariance of $d(Q_e, Q_m)$ under the duality transformation:

$$\begin{pmatrix} Q_m/\sqrt{2} \\ \sqrt{2}Q_e \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q_m/\sqrt{2} \\ \sqrt{2}Q_e \end{pmatrix} \tag{4.19}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2)$, i.e.

$$ad - bc = 1, \quad a, d \in 1 + 2\mathbb{Z}, \quad c \in 2\mathbb{Z}, \quad b \in \mathbb{Z}\,. \tag{4.20}$$

These transformation laws are somewhat different in appearance from the standard duality transformation laws discussed e.g. in [6]. This is due to the fact that the degeneracy formula (3.21) is related to the corresponding formula in [6] by the transformation $Q_e^2 \to Q_m^2/2$, $Q_m^2 \to 2Q_e^2$. However eqs. (4.19), (4.20) can be reexpressed in the form:

$$\begin{pmatrix} Q_e \\ Q_m \end{pmatrix} \to \begin{pmatrix} d & c/2 \\ 2b & a \end{pmatrix} \begin{pmatrix} Q_e \\ Q_m \end{pmatrix}\,, \tag{4.21}$$

with $\begin{pmatrix} d & c/2 \\ 2b & a \end{pmatrix} \in \Gamma_1(2)$. This is the usual form of S-duality transformation in the second description of the system.

4.3 Location of the zeroes of $\tilde{\Phi}$

We can follow the procedure of [9] to identify the location of the zeroes of $\tilde{\Phi}$ by examining the location of the singularities in the integral $\mathcal{I}$. As in [9] one finds that $\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ has possible zeroes at

$$\begin{aligned} &\left(n_2(\tilde{\sigma}\tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho}m_1 + m_2\right) = 0 \\ &\quad \text{for } m_1, m_2, n_2 \in \mathbb{Z},\ n_1 \in \tfrac{1}{2}\mathbb{Z},\ b \in 2\mathbb{Z} + 1, \quad m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = \frac{1}{4}\,. \end{aligned} \tag{4.22}$$

The order of the zero is given by

$$\sum_{s=0}^{1} (-1)^{m_1 s} c^{(r,s)}(-1), \qquad r = 2n_1 \bmod 1\,. \tag{4.23}$$

Using (4.10) we see that (4.23) vanishes for $r = 1$. Thus in order to get a zero (or pole), n_1 must be an integer. Setting $r = 0$ in (4.23) and using (4.10) we see that the order of the zero is now given by $2 \times (-1)^{m_1}$. Thus $\tilde{\Phi}$ has second order zeroes at

$$\begin{aligned}&\big(n_2(\tilde{\sigma}\tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho}m_1 + m_2\big) = 0\\ &\quad \text{for } m_1 \in 2\mathbb{Z},\ m_2, n_2 \in \mathbb{Z},\ n_1 \in \mathbb{Z},\ b \in 2\mathbb{Z}+1, \quad m_1n_1 + m_2n_2 + \frac{b^2}{4} = \frac{1}{4}\,,\end{aligned} \tag{4.24}$$

and second order poles at

$$\begin{aligned}&\big(n_2(\tilde{\sigma}\tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho}m_1 + m_2\big) = 0\\ &\quad \text{for } m_1 \in 2\mathbb{Z}+1,\ m_2, n_2 \in \mathbb{Z},\ n_1 \in \mathbb{Z},\ b \in 2\mathbb{Z}+1, \quad m_1n_1 + m_2n_2 + \frac{b^2}{4} = \frac{1}{4}\,.\end{aligned} \tag{4.25}$$

We shall now determine the constant of proportionality for two particular cases, namely near $\tilde{v} = 0$ and near $\tilde{\sigma}\tilde{\rho} - \tilde{v}^2 + \tilde{v} = 0$. The $\tilde{v} \to 0$ behaviour of $\tilde{\Phi}$ can be derived directly from (3.19) and the relations

$$\sum_b c^{(r,s)}(4n - b^2) = \begin{cases} 0 & \text{for} \quad (r,s) = (0,0) \\ 8\,\delta_{n,0} & \text{for} \quad (r,s) \neq (0,0) \end{cases}\,, \tag{4.26}$$

which follow from setting $z = 0$ in eqs. (3.10) and (3.14). This gives

$$\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) \simeq \frac{\pi^2}{64}\,\tilde{v}^2 \frac{\eta(2\tilde{\rho})^{16}}{\eta(\tilde{\rho})^8}\frac{\eta(\tilde{\sigma}/2)^{16}}{\eta(\tilde{\sigma})^8}\,. \tag{4.27}$$

In order to find the behaviour of $\tilde{\Phi}$ near $\tilde{\sigma}\tilde{\rho} - \tilde{v}^2 + \tilde{v} = 0$ we first note from (4.15) that for $v \to 0$

$$\Phi(\rho, \sigma, v) \simeq 4\pi^2 v^2 \frac{\eta(2\rho)^{16}}{\eta(\rho)^8}\frac{\eta(2\sigma)^{16}}{\eta(\sigma)^8} + \mathcal{O}(v^4)\,. \tag{4.28}$$

Next we use the duality transformation property

$$\Phi(\rho, \sigma + 2v + \rho, v + \rho) = \Phi(\rho, \sigma, v)\,, \tag{4.29}$$

which follows from the symmetry of $\mathcal{I}'$ under a relabelling of the indices $b, \vec{m}, \vec{n}$ in eq. (4.13). Eqs. (4.16) and (4.29) give

$$\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \tilde{\sigma}^{-2}\,\Phi\left(\tilde{\rho} - \frac{\tilde{v}^2}{\tilde{\sigma}}, \frac{\tilde{\rho}\tilde{\sigma} - (\tilde{v}-1)^2}{\tilde{\sigma}}, \frac{\tilde{\sigma}\tilde{\rho} - \tilde{v}^2 + \tilde{v}}{\tilde{\sigma}}\right)\,. \tag{4.30}$$

(4.28) now gives, for small $\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v}$,

$$\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = 4\pi^2\,(2v - \rho - \sigma)^2\,v^2\,f(\rho)f(\sigma) + \mathcal{O}(v^4)\,, \tag{4.31}$$

where

$$f(\rho) = \eta(2\rho)^{16}/\eta(\rho)^8\,, \tag{4.32}$$

and

$$\rho = \frac{\tilde\rho\tilde\sigma - \tilde v^2}{\tilde\sigma}, \qquad \sigma = \frac{\tilde\rho\tilde\sigma - (\tilde v - 1)^2}{\tilde\sigma}, \qquad v = \frac{\tilde\rho\tilde\sigma - \tilde v^2 + \tilde v}{\tilde\sigma}, \tag{4.33}$$

or equivalently,

$$\tilde\rho = \frac{v^2 - \rho\sigma}{2v - \rho - \sigma}, \qquad \tilde\sigma = \frac{1}{2v - \rho - \sigma}, \qquad \tilde v = \frac{v - \rho}{2v - \rho - \sigma}. \tag{4.34}$$

These relations will be useful in section 5 for evaluating the statistical entropy of the black hole.

5. Statistical and black hole entropy functions

In this section we shall compute the statistical entropy function [9] of the dyons carrying electric charge Q_e and magnetic charge Q_m. The value of this function at its extremum gives the statistical entropy, — the logarithm of the degeneracy of states corresponding to a given set of charges. We also compute the black hole entropy function [29, 30] whose value at its extremum gives the Wald entropy of the black hole. We then compare the two results.

5.1 Statistical entropy function

We begin with the formula (3.21) for the degeneracy of dyons:

$$d(Q_e, Q_m) = K \int_C d\tilde\rho d\tilde\sigma d\tilde v \frac{1}{\tilde\Phi(\tilde\rho, \tilde\sigma, \tilde v)} \exp\left[-i\pi(2\tilde\rho Q_e^2 + \tilde\sigma Q_m^2/2 + 2\tilde v Q_e\cdot Q_m)\right]. \tag{5.1}$$

This formula is identical in form to eq. (3.29) of [9] with the substitution $Q_m^2 \to 2Q_e^2$, $Q_e^2 \to Q_m^2/2$. Following [1, 9] one can show that the dominant contribution to this integral comes from the residue at the pole at

$$\tilde\sigma\tilde\rho - \tilde v^2 + \tilde v = 0. \tag{5.2}$$

The behaviour of $\tilde\Phi$ near this zero, given by (4.31), is identical to the corresponding relation (4.17) in [9] with $k \to 2$ and $f^{(k)}(\rho) \to f(\rho)$. Thus following an analysis identical to that in [9] we can conclude that for large charges the statistical entropy $S_{stat}(Q_e, Q_m)$, defined as the logarithm of the degeneracy $d(Q_e, Q_m)$, is obtained by extremizing the statistical entropy function

$$-\tilde\Gamma_B(\vec\tau') = \frac{\pi}{2\tau_2'}\,|\frac{1}{\sqrt2}Q_m + \sqrt2\,\tau'\,Q_e|^2 - \ln f(\tau') - \ln f(-\bar\tau') - 4\ln(2\tau_2') + \text{constant} + \mathcal{O}(Q^{-2}). \tag{5.3}$$

with respect to the real and imaginary parts of τ'. In terms of a new variable

$$\tau = \frac{1}{2\bar\tau'}, \tag{5.4}$$

we can express (5.3) as

$$-\tilde\Gamma_B = \frac{\pi}{2\tau_2}\,|Q_e + \tau Q_m|^2 - \ln\tilde f(\tau) - \ln\tilde f(-\bar\tau) - 4\ln(2\tau_2) + \text{constant} + \mathcal{O}(Q^{-2}), \tag{5.5}$$

where

$$\tilde{f}(\tau) = \eta(\tau)^{16}/\eta(2\tau)^8 \,. \tag{5.6}$$

For large charges the first term on the right hand side of (5.5) gives the leading contribution to the statistical entropy. This term is universal and coincides e.g. with the corresponding term in the statistical entropy function for CHL models. The rest of the terms, giving correction of order Q^0 or lower to the entropy, depend on the specific theory being analyzed.

5.2 Black hole entropy function

As discussed at the end of section 2, the low energy effective field theory describing the theory under consideration is $\mathcal{N}=4$ supergravity coupled to six matter multiplets. Since the supergravity action is insensitive to the details of the theory except for the rank of the gauge group, the Bekenstein-Hawking entropy of a BPS black hole carrying charges (Q_e, Q_m), computed using the supergravity action, reproduces the leading contribution of order Q^2 to the statistical entropy as in the case of toroidally compactified heterotic string theory or CHL models. However since we shall be interested in computing the entropy to order Q^0 we must also include four derivative corrections to the supergravity action. An important set of four derivative terms relevant for computing the order Q^0 corrections to the entropy is the Gauss-Bonnet term. For definiteness we shall use the second description of the theory to describe these corrections. On general grounds the Gauss-Bonnet term can be shown to have the following structure[5]

$$\Delta\mathcal{L} = \phi(a,S)\,\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right\}\,, \tag{5.7}$$

where $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R are the Riemann tensor, Ricci tensor and scalar curvatures respectively, $S = e^{-2\Phi}$ where Φ is the dilaton field and a is the axion field obtained by dualizing the rank two anti-symmetric tensor field in four dimensions. The function $\phi(a,S)$ has the structure:

$$\phi(a,S) = -\frac{1}{128\pi^2}\left(\mathcal{K}\ln(2\,S) + g(a+iS) + g(a+iS)^*\right) \tag{5.8}$$

where $\mathcal{K}$ is a constant representing the effect of holomorphic anomaly [31, 32], and $g(\tau)$ is a holomorphic function of τ which will be determined shortly. Explicit result for $\phi(a,S)$ for this model can be found in [33], but we shall describe an alternative method for determining $\phi(a,S)$ following [34] which can be easily generalized to the case of $\mathbb{Z}_3$ orbifold to be discussed in section 6. $\phi(a,S)$ is invariant under the S-duality group $\Gamma_1(2)$, which acts on $\tau \equiv a+iS \equiv \tau_1 + i\tau_2$ as

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad a,b,c,d \in \mathbb{Z}, \quad ad-bc=1, \quad c=0 \text{ mod } 2, \quad a,d=1 \text{ mod } 2\,. \tag{5.9}$$

Thus gives

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = g(\tau) + \mathcal{K}\ln(c\tau+d)\,, \tag{5.10}$$

[5]There is also a term proportional to the imaginary part of the function $g(a+iS)$ multiplying the Pontryagin density. But this term does not play any role in the analysis of the entropy of spherically symmetric black holes since its contribution to the black hole entropy function vanishes.

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and hence

$$g(\tau) - 2\,\mathcal{K} \ln \eta(\tau) \tag{5.11}$$

is invariant under a modular transformation except for a constant shift originating from the phases picked up by $\eta(\tau)$ under a modular transformation. Thus

$$\partial_\tau \left(g(\tau) - 2\,\mathcal{K} \ln \eta(\tau) \right) \tag{5.12}$$

must be a modular form of $\Gamma_1(2)$ of weight 2. There is a unique modular form with this property [35], namely

$$\partial_\tau \left(\ln \eta(2\tau) - \ln \eta(\tau) \right) \ . \tag{5.13}$$

Thus (5.12) must be proportional to (5.13). The constant of proportionality may be determined as follows. Since toroidally compactified type II string theory has no Gauss-Bonnet term at the tree level, such terms are absent even after taking the orbifold projection. This shows that $\phi(a, S)$, and hence $g(a + iS)$ cannot have a term growing linearly with S for large S. Comparing the large S behaviour of (5.12) and (5.13) we now get

$$g(\tau) - 2\,\mathcal{K} \ln \eta(\tau) = -2\,\mathcal{K} \left(\ln \eta(2\tau) - \ln \eta(\tau) \right) + \text{constant} \,, \tag{5.14}$$

or equivalently

$$g(\tau) = -2\,\mathcal{K} \left(\ln \eta(2\tau) - 2 \ln \eta(\tau) \right) + \text{constant} \,. \tag{5.15}$$

This gives

$$\begin{aligned} \phi(a, S) \, = \, & -\frac{\mathcal{K}}{128\pi^2} \left(\ln(2\,\tau_2) - 2\left(\ln \eta(2\tau) - 2 \ln \eta(\tau) \right) - 2\left(\ln \eta(-2\bar\tau) - 2 \ln \eta(-\bar\tau) \right) \right) \\ & + \text{constant} \,. \end{aligned} \tag{5.16}$$

Finally we turn to the determination of $\mathcal{K}$. This is done following the procedure described in [34] with K3 replaced by T^4. The net result is that $\mathcal{K}$ is the number of harmonic p forms on T^4 invariant under the transformation $\tilde g$, weighted by $(-1)^p$. Since only the even forms are invariant under $\tilde g$, and there are altogether 8 even forms on T^4, we get

$$\mathcal{K} = 8 \,. \tag{5.17}$$

This determines the structure of the Gauss-Bonnet term completely. The result agrees with the result of explicit computation described in [33].

The effect of the term given in (5.7) on the computation of black hole entropy was analyzed in detail in [30]. After elimination of all variables except the values of a and S on the horizon, the black hole entropy function takes the form:

$$\begin{aligned} \mathcal{E} \, &= \, \frac{\pi}{2\tau_2} \left| Q_e + \tau Q_m \right|^2 + 64\,\pi^2\, \phi(\tau_1, \tau_2) \\ &= \frac{\pi}{2\tau_2} \left| Q_e + \tau Q_m \right|^2 - 4 \ln(2\tau_2) + 8 \left(\ln \eta(2\tau) - 2 \ln \eta(\tau) \right) + 8 \left(\ln \eta(-2\bar\tau) - 2 \ln \eta(-\bar\tau) \right) \\ & \quad + \text{constant} \,. \end{aligned} \tag{5.18}$$

Extremization of this function with respect to τ_1 and τ_2 gives the black hole entropy. Comparing (5.5) and (5.18) we see that the black hole entropy and the statistical entropy agree to this order.[6]

Given that for this model the black hole entropy fails to agree with the statistical entropy for elementary string states [21, 22], it is worth trying to understand the difference between these two cases. First we note that if we take $Q_e^2 \gg Q_m^2, (Q_e \cdot Q_m)^2/Q_m^2$ in the expression (5.18) for the black hole entropy function, then extremization of the first term requires τ_2 to be large. In this limit the term growing linearly with τ_2 in the rest of the terms cancel. This does not happen for the corresponding expression ((4.41) in [9]) for the black hole entropy function for the CHL models. Thus although the leading contribution to the black hole entropy is the same in all $\mathcal{N} = 4$ supersymmetric compactifications, the correction to this leading term is smaller in the present model compared to the CHL models by powers of Q_m^2/Q_e^2 and $(Q_e \cdot Q_m)^2/Q_e^2 Q_m^2$. This of course is a consequence of the absence of tree level curvature squared corrections in type II string theory.

How does this difference come about in the formula for the statistical entropy? For this we need to understand the origin of the corrections linear in τ in the statistical entropy function. Let us for definiteness work in the second description of the model where purely electrically charged states represent elementary string states. On physical grounds we should expect that when the electric charge is large compared to the magnetic charge the correction to the leading contribution to the statistical entropy will be dominated by the growth in the degeneracy of elementary string states, i.e. the contribution (3.5) in the present model or its analog in the case of CHL models (eq. (3.2) of [9]). This intuition can be put on a firmer ground by noting that if we remove this factor from the dyon partition function then the modified statistical entropy, computed using this modified partition function, does not contain any term growing linearly with τ, either in the present model or in the CHL models. Thus the term in the entropy function growing linearly with τ has its origin in the partition function of elementary string states, and the difference in the behaviour of the statistical entropy function in the present model and the CHL models can be attributed to a difference in behaviour of the elementary string partition function in the two theories.

By examining carefully the analysis of [9] leading to the final expression for the statistical entropy function one can check that the large τ behaviour of the correction term is controlled by the small $\tilde{\rho}$ behaviour of the partition function (3.5) of elementary string states. In particular the absence of linear corrections to the statistical entropy function of the present model is related to the absence of exponential divergence of (3.5) in the $\tilde{\rho} \to 0$

[6] We should remind the reader that the string theory effective action has other four derivative terms besides the one given in (5.7) and hence regarding (5.18) as the complete contribution to the black hole entropy function to this order is not completely justified. A somewhat different set of four derivative terms, based on supersymmetrization of the curvature squared terms, give the same answer for the black hole entropy [36, 37]. Thus it seems that the answer for the black hole entropy, obtained by extremizing (5.18), is somewhat robust. Nevertheless it will be useful to determine the complete set of four derivative corrections to the supergravity action and study their effect on the black hole entropy. An attempt towards this has recently been made in [38].

limit. In contrast the corresponding elementary string partition function for (say) the $\mathbb{Z}_2$ CHL model has the form [9]:

$$\eta(\tilde{\rho})^{-8}\eta(2\tilde{\rho})^{-8}\,, \tag{5.19}$$

and diverges exponentially as $\tilde{\rho} \to 0$. This difference in behaviour might seem a bit surprising at the first sight since the small $\tilde{\rho}$ behaviour of the partition function controls the growth of degeneracy for large charges and for both models the degeneracy grows exponentially. The difference however comes from the fact that (3.5) and (5.19) actually represent an index where we multiply the degeneracy by $(-1)^{F_L}$, F_L being the space-time fermion number associated with left-moving world-sheet excitations. For the CHL model all the left-moving excitations are bosonic and hence the degeneracy is equal to this index. The exponential growth in the degeneracy causes an exponential divergence in the partition function (5.19) as $\tilde{\rho} \to 0$. However for the present model, states with even and odd momentum along S^1 correspond to bosonic and fermionic states respectively [21, 22], and the index is equal to the degeneracy up to a sign. The small $\tilde{\rho}$ behaviour of the 'partition function' (3.5) is controlled by the difference in the growth rate between bosonic and fermionic excitations and the leading exponential term cancels between these set of states. As a result (3.5) has no exponential divergence in the $\tilde{\rho} \to 0$ limit.

To summarize the situation, we have seen that the absence/presence of linearly growing correction to the statistical entropy function in the present/CHL model can be attributed to the fact that in the present model elementary string spectrum contains both bosonic and fermionic excitations in the left-moving sector, whereas the CHL model has only bosonic excitations in the left-moving sector. Nevertheless this by itself would not provide a complete physical explanation of the difference in behaviour of the statistical entropy functions in the two theories since the statistical entropy is computed for a fixed charge, and the elementary string states with bosonic and fermionic left-moving excitations carry different charges.[7] We must recall however that the complete description of a state of the dyon involves a tensor product of states from three different Hilbert spaces. Thus for example a fermionic elementary string state carrying odd momentum along S^1 combined with an odd momentum state from another sector and a bosonic elementary string state carrying even momentum along S^1, combined with an even momentum state from another sector, can give rise to states carrying the same charge but opposite statistics. Their net contribution to the index will then be zero, causing a suppression in the statistical entropy function. Such cancellations will not take place in the corresponding CHL models.

This seems to be the physical explanation for why for dyonic states the linearly growing corrections to the statistical entropy function are absent in the present model in agreement with the black hole entropy, while for the statistical entropy of elementary string states there are no such cancellations between bosonic and fermionic states.

[7] This in fact is the reason why, just as in CHL models, the statistical entropy of an elementary string state still grows exponentially in this theory in disagreement with the black hole entropy.

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6. The $\mathbb{Z}_3$ orbifold example

In this section we shall analyze the dyon spectrum in another $\mathcal{N} = 4$ supersymmetric theory, obtained by taking a $\mathbb{Z}_3$ orbifold of type IIA string theory compactified on a six torus $T^4 \times S^1 \times \tilde{S}^1$. The orbifold group involves a $2\pi/3$ rotation along one two dimensional plane in T^4, $-2\pi/3$ rotation along an orthogonal two dimensional plane in T^4 and $1/3$ unit of shift along the circle S^1. This of course requires that the $\mathbb{Z}_3$ transformation is a symmetry of the original torus T^4, — this can be achieved for example by taking T^4 to be a product of two two dimensional tori, each with a hexagonal symmetry. There is a dual description of these models, also as orbifolds of type IIA string theory on a six torus $\hat{T}^4 \times S^1 \times \hat{S}^1$, but now the orbifold group involves $1/3$ unit of shift along S^1 together with a rotation by $4\pi/3$ in a coordinate plane in $\hat{T}^4$ acting only on the left-moving world-sheet fields [20]. As in the case of $\mathbb{Z}_2$ orbifold model, this theory also has $\mathcal{N} = 4$ supersymmetry in four dimensions. The gauge group now has rank 10 since (in the NSR formulation) besides all the RR sector gauge fields, two of the gauge fields originating in the NS-NS sector are also projected out in the second description. The S-duality group in the second description is $\Gamma_1(3)$, consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying

$$ad - bc = 1, \quad a, d \in 1 + 3\mathbb{Z}, \quad c \in 3\mathbb{Z}, \quad b \in \mathbb{Z}\,. \tag{6.1}$$

The various parts of the analysis done for the $\mathbb{Z}_2$ orbifold model can be easily generalized to the case of this $\mathbb{Z}_3$ orbifold model by following [7, 9]. For the sake of brevity we shall not repeat the analysis here but only give the final results. Also in order to make the comparison between the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ models easier we shall state the results for $\mathbb{Z}_N$ model which will be valid both for $N = 2$ and $N = 3$. Thus by setting $N = 2$ we can recover the results of the previous sections.

First of all we note that in both models the rank r of the gauge group may be expressed as

$$r = 2k + 8\,, \tag{6.2}$$

where

$$k + 2 = \frac{12}{N+1}\,. \tag{6.3}$$

The degeneracy formula takes the form

$$d(Q_e, Q_m) = K \int_C d\tilde{\rho}\, d\tilde{\sigma}\, d\tilde{v} \frac{1}{\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})} \exp\left[-i\pi(N\tilde{\rho}Q_e^2 + \tilde{\sigma}Q_m^2/N + 2\tilde{v}Q_e \cdot Q_m)\right]\,, \tag{6.4}$$

where

$$K = -N^{-1-N(k+2)/(N-1)}\,, \tag{6.5}$$

C is the hypersurface

$$\begin{aligned} &Im\,\tilde{\rho} = M_1, \quad Im\,\tilde{\sigma} = M_2, \quad Im\,\tilde{v} = M_3, \\ &0 \le Re\,\tilde{\rho} \le 1, \quad 0 \le Re\,\tilde{\sigma} \le N, \quad 0 \le Re\,\tilde{v} \le 1\,, \end{aligned} \tag{6.6}$$

and

$$\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v}) = -N^{-N(k+2)/(N-1)}\, e^{2\pi i(\tilde{\rho}+\tilde{v})} \times \prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},l,j\in\mathbb{Z}\\ k',l\geq 0, j<0 \text{ for } k'=l=0}} \left(1-e^{2\pi i(\tilde{\sigma}k'+\tilde{\rho}l+\tilde{v}j)}\right)^{\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(r,s)}(4lk'-j^2)} . \tag{6.7}$$

The coefficients $c^{(r,s)}(4lk'-j^2)$ are given as follows. Let us define

$$F^{(r,s)}(\tau,z) \equiv \frac{1}{N} Tr_{RR;\tilde{g}^r}\left(\tilde{g}^s(-1)^{F_L+F_R} e^{2\pi i\tau L_0} e^{2\pi i\mathcal{J}z}\right), \qquad r,s=0,1,\dots N-1\,, \tag{6.8}$$

where $\tilde{g}$ denotes the part of the orbifold action in the first description that acts as rotation by angles $(2\pi/N,-2\pi/N)$ on the two orthogonal planes of a four torus T^4, and the trace is taken over all the RR sector states twisted by $\tilde{g}^r$ in the $\mathbb{Z}_N$ orbifold of the (4,4) superconformal field theory with target space T^4, – with $\mathbb{Z}_N$ generated by $\tilde{g}$, — before we project on to $\tilde{g}$ invariant states. F_L and F_R denote the world-sheet fermion numbers associated with left and right chiral fermions in this SCFT, and $\mathcal{J}/2$ is the generator of the $\mathrm{U}(1)_L$ subgroup of the $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ R-symmetry group of this conformal field theory. One finds that $F^{(r,s)}(\tau,z)$ has expansion of the form

$$F^{(r,s)}(\tau,z) = \sum_{b\in\mathbb{Z},n\in\mathbb{Z}/N} c^{(r,s)}(4n-b^2) e^{2\pi in\tau+2\pi ibz}\,. \tag{6.9}$$

This defines the coefficients $c^{(r,s)}(4n-b^2)$.

The explicit forms of $F^{(r,s)}(\tau,z)$ are as follows

$$\begin{aligned} F^{(0,s)}(\tau,z) &= \frac{16}{N}\sin^4\left(\frac{\pi s}{N}\right) \frac{\vartheta_1\left(\tau,z+\frac{s}{N}\right)\vartheta_1\left(\tau,-z+\frac{s}{N}\right)}{\vartheta_1\left(\tau,\frac{s}{N}\right)^2} \\ F^{(r,s)}(\tau,z) &= \frac{4\,N}{(N-1)^2} \frac{\vartheta_1\left(\tau,z+\frac{s}{N}+\frac{r}{N}\tau\right)\vartheta_1\left(\tau,-z+\frac{s}{N}+\frac{r}{N}\tau\right)}{\vartheta_1\left(\tau,\frac{s}{N}+\frac{r}{N}\tau\right)^2}\,, \\ &\text{for } 1\leq r\leq N-1,\ 0\leq s\leq N-1\,. \end{aligned} \tag{6.10}$$

A factor of $4\sin^2\left(\frac{\pi s}{N}\right)$ in the expression for $F^{(0,s)}(\tau,z)$ comes from the contribution due to the right-moving fermionic zero modes. A factor of $\frac{4\,N^2}{(N-1)^2}$ in the expression for $F^{(r,s)}$ counts the number of twisted sectors. Using standard identities involving Jacobi ϑ-functions we may rewrite (6.10) as

$$F^{(r,s)}(\tau,z) = h_0^{(r,s)}(\tau)\vartheta_3(2\tau,2z) + h_1^{(r,s)}(\tau)\vartheta_2(2\tau,2z)\,, \tag{6.11}$$

where

$$\begin{aligned} h_0^{(0,s)}(\tau) &= -\frac{16}{N}\sin^4\frac{\pi s}{N}\frac{1}{\vartheta_1\left(\tau,\frac{s}{N}\right)^2}\vartheta_2\left(2\tau,2\frac{s}{N}\right)\,, \\ h_1^{(0,s)}(\tau) &= \frac{16}{N}\sin^4\frac{\pi s}{N}\frac{1}{\vartheta_1\left(\tau,\frac{s}{N}\right)^2}\vartheta_3\left(2\tau,2\frac{s}{N}\right)\,, \end{aligned}$$

$$
\begin{aligned}
h_0^{(r,s)}(\tau) &= -\frac{4N}{(N-1)^2}\,\frac{1}{\vartheta_1\left(\tau,\frac{1}{N}(s+r\tau)\right)^2}\,\vartheta_2\left(2\tau,\frac{2}{N}(s+r\tau)\right)\,,\\
h_1^{(r,s)}(\tau) &= \frac{4N}{(N-1)^2}\,\frac{1}{\vartheta_1\left(\tau,\frac{1}{N}(s+r\tau)\right)^2}\,\vartheta_3\left(2\tau,\frac{2}{N}(s+r\tau)\right)\,,\\
&\qquad 0\le s\le (N-1), \quad 1\le r\le (N-1)\,.
\end{aligned}
\tag{6.12}
$$

The coefficients $c^{(r,s)}(u)$ may now be defined through the expansion

$$
h_l^{(r,s)}(\tau) = \sum_{n\in\frac{1}{N}\mathbb{Z}-\frac{l}{4}} c^{(r,s)}(4n)e^{2\pi i n\tau}\,. \tag{6.13}
$$

From (6.13) one can calculate the coefficients $c^{(r,s)}(u)$ explicitly.[8]

Generalizing the analysis of section 4 one can show that the function $\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})$ transforms as a modular form of weight k under a certain subgroup of the Siegel modular group of genus two Riemann surfaces, with k given by (6.3). This subgroup is the same one that appears in the analysis of [6, 7] for $\mathbb{Z}_N$ CHL model. Using this fact one can prove that the degeneracy formula (6.4) is invariant under the S-duality group $\Gamma_1(N)$ of the theory.

Analysis of the behaviour of the statistical entropy for large charges shows that this is given by extremizing a statistical entropy function

$$
\frac{\pi}{2\tau_2}\,|Q_e+\tau Q_m|^2 - \ln\tilde{f}_k(\tau) - \ln\tilde{f}_k(-\bar{\tau}) - (k+2)\ln(2\tau_2) + \text{constant} + \mathcal{O}(Q^{-2})\,, \tag{6.15}
$$

with respect to the real and imaginary parts of $\tau=\tau_1+i\tau_2$. Here

$$
\tilde{f}_k(\tau) = \eta(\tau)^{2N(k+2)/(N-1)}\eta(N\tau)^{-2(k+2)/(N-1)}\,. \tag{6.16}
$$

In order to compute the black hole entropy we need to determine the function $\phi(a,S)$ introduced in (5.7). This can be done by generalizing the analysis of section 5.2; all that changes is that (5.13) now takes the form $\partial_\tau\left(\ln\eta(N\tau)-\ln\eta(\tau)\right)$ and $\mathcal{K}$ is given by $2k+4$. The result is

$$
\begin{aligned}
\phi(a,S) = &-\frac{k+2}{64\pi^2}\bigg(\ln(2\,\tau_2) - \frac{2}{N-1}\left(\ln\eta(N\tau)-N\ln\eta(\tau)\right)\\
&-\frac{2}{N-1}\left(\ln\eta(-N\bar{\tau})-N\ln\eta(-\bar{\tau})\right)\bigg) + \text{constant}
\end{aligned}
\tag{6.17}
$$

Using this the black hole entropy function becomes

$$
\frac{\pi}{2\tau_2}\,|Q_e+\tau Q_m|^2 - \ln\tilde{f}_k(\tau) - \ln\tilde{f}_k(-\bar{\tau}) - (k+2)\ln(2\tau_2) + \text{constant}\,. \tag{6.18}
$$

Thus again we see that the black hole entropy and the statistical entropy match to this order.

---

[8] Incidentally, the coefficients $c^{(r,s)}(u)$ are related to the corresponding coefficients for the $\mathbb{Z}_N$ CHL model (which we shall denote by $c_{chl}^{(r,s)}(u)$) via the relations

$$
c^{(r,s)}(u) = \begin{cases} 0 \quad \text{for} \quad (r,s)=(0,0) \\ \frac{N}{N-1}\,c_{chl}^{(r,s)}(u) \quad \text{for} \quad (r,s)\neq(0,0)\,. \end{cases} \tag{6.14}
$$

Finally, to complete the comparison with the corresponding analysis for the CHL orbifold models, we note that it is possible to find a series formula for the modular form $\tilde{\Phi}$ and its closely related cousin Φ defined through

$$\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v}) = -(-i\tilde{\sigma})^{-k}\,\Phi\left(\tilde{\rho}-\frac{\tilde{v}^2}{\tilde{\sigma}}, -\frac{1}{\tilde{\sigma}}, \frac{\tilde{v}}{\tilde{\sigma}}\right), \tag{6.19}$$

by repeating the analysis of [6]. This is done by replacing the cusp form $f^{(k)}(\tau)$ used in [6] by the modular form

$$f_k(\tau) = \eta(N\tau)^{2N(k+2)/(N-1)}\eta(\tau)^{-2(k+2)/(N-1)} \tag{6.20}$$

of $\Gamma_1(N)$ of weight $(k+2)$. Both for $N=2$ and $N=3$, $f_k(\tau)$ vanishes as $q=e^{2\pi i\tau}$ at the cusp at $\tau\to i\infty$. However using the modular transformation properties of $\eta(\tau)$ it is easy to see that $\tau^{-k-2}f_k(-1/\tau)$ goes to a constant as $\tau\to i\infty$. Thus $f_k(\tau)$ is not a cusp form of $\Gamma_1(N)$. Nevertheless we can proceed as in [6] to construct (meromorphic) modular forms Φ and $\tilde{\Phi}$ of weight k of appropriate subgroups of $\mathrm{Sp}(2,\mathbb{Z})$. For example the modular form Φ is given by a formula analogous to eq. (1.6) of [6]

$$\Phi(\rho,\sigma,v) = \sum_{\substack{n,m,r\in\mathbb{Z}\\ n,m\geq 1,\, r^2<4mn}} a(n,m,r)\, e^{2\pi i(n\rho+m\sigma+rv)}, \tag{6.21}$$

where,

$$a(n,m,r) = \sum_{\substack{\alpha\in\mathbb{Z};\alpha>0\\ \alpha|(n,m,r),\ \text{g.c.d.}(\alpha,N)=1}} \chi(\alpha)\,\alpha^{k-1}\,C\left(\frac{4mn-r^2}{\alpha^2}\right), \tag{6.22}$$

$$\begin{aligned}\chi(\alpha) &= 1 \quad \text{for } N=2\\ &= \begin{cases}1 & \text{for } \alpha = 1 \bmod 3\\ -1 & \text{for } \alpha = 2 \bmod 3\end{cases} \quad \text{for } N=3\,.\end{aligned} \tag{6.23}$$

The coefficients $C(m)$ are obtained from the modular form $f_k(\tau)$ as follows. We first define the coefficients $f_{k,n}$ as

$$f_k(\tau)\eta(\tau)^{-6} = \sum_{n\geq 1} f_{k,n} e^{2\pi i\tau(n-\frac{1}{4})}, \tag{6.24}$$

and then define $C(m)$ as

$$C(m) = (-1)^m \sum_{\substack{s,n\in\mathbb{Z}\\ n\geq 1}} f_{k,n}\delta_{4n+s^2-1,m}\,. \tag{6.25}$$

Eq. (6.21) gives a series expansion for Φ. A similar series expansion for $\tilde{\Phi}$ may be found by following the analysis of [6] (see eq. (C.37) of this paper) but since the formulæ are considerably more complicated we shall not describe it here.

Acknowledgments

We would like to thank E. Gava and K.S. Narain for useful discussions.

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